P Preparation for Calculus

P.3 Functions and Their Graphs

Objectives

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.
- Sketch the graph of a function.
- Identify different types of transformations of functions.
- Classify functions and recognize combinations of functions.

Functions and Function Notation

Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y.

A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value.

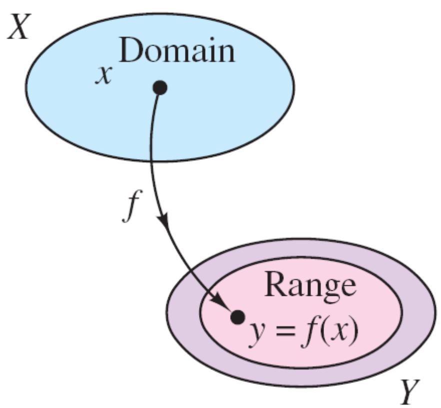
The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Functions and Function Notation

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X. (See Figure P.22.)



A real-valued function f of a real variable

Figure P.22

Functions and Function Notation

Functions can be specified in a variety of ways. However, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines *y*, the dependent variable, as a function of *x*, the independent variable.

Functions and Function Notation

To **evaluate** this function (that is, to find the *y*-value that corresponds to a given *x*-value), it is convenient to isolate *y* on the left side of the equation.

$$y = \frac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using *f* as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation, $x^2 + 2y = 1$, **implicitly** defines *y* as a function of *x*. When you solve the equation for *y*, you are writing the equation in **explicit** form.

Example 1 – *Evaluating a Function*

For the function *f* defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Solution:

a. $f(3a) = (3a)^2 + 7$ Substitute $3a$ for x.

$= 9a^2 + 7$ Simplify.

b. $f(b - 1) = (b - 1)^2 + 7$ Substitute $b - 1$ for x.

$= b^2 - 2b + 1 + 7$ Expand binomial.

$= b^2 - 2b + 8$ Simplify.

Example 1 – *Solution*

cont'd

c. $$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{[(x+\Delta x)^2+7]-(x^2+7)}{\Delta x}$$

$$=\frac{x^2+2x\Delta x+(\Delta x)^2+7-x^2-7}{\Delta x}$$

$$=\frac{2x\Delta x+(\Delta x)^2}{\Delta x}$$

$$=\frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

$$=2x+\Delta x, \quad \Delta x \neq 0$$

The Domain and Range of a Function

The Domain and Range of a Function

The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function.

The **implied domain** is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function.

For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$.

The Domain and Range of a Function

On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x: x \neq \pm 2\}$.

Example 2 – *Finding the Domain and Range of a Function*

Find the domain and range of each function.

a. $f(x) = \sqrt{x - 1}$ **b.** $g(x) = \sqrt{4 - x^2}$

Solution:

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

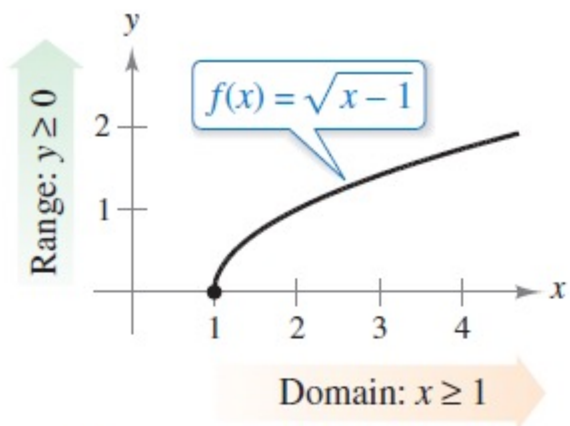
is the set of all *x*-values for which $x - 1 \geq 0$, which is the interval $[1, \infty)$.

To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative.

Example 2 – *Solution*

cont'd

So, the range is the interval $[0, \infty)$, as shown in Figure P.23(a).



(a) The domain of f is $[1, \infty)$, and the range is $[0, \infty)$.

Figure P.23

Example 2 – *Solution*

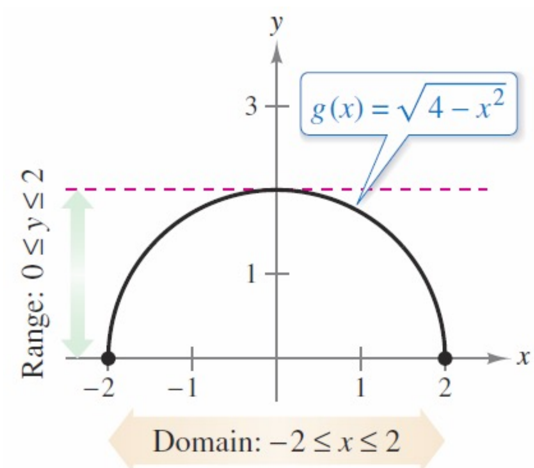
cont'd

b. The domain of the function

$$g(x) = \sqrt{4 - x^2}$$

is the set of all values for which $4 - x^2 \geq 0$, or $x^2 \leq 4$. So, the domain of g is the interval [–2, 2].

To find the range, observe that $g(x) = \sqrt{4 - x^2}$ is never negative and is at most 2. So, the range is the interval [0, 2], as shown in Figure P.23(b).



(b) The domain of g is $[-2, 2]$, and the range is $[0, 2]$.

Figure P.23

The Domain and Range of a Function

A function from X to Y is **one-to-one** when to each y-value in the range there corresponds exactly one x-value in the domain.

A function from X to Y is **onto** when its range consists of all of Y.

The Graph of a Function

The Graph of a Function

The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure P.25, note that

x = the directed distance from the y-axis

and

$f(x)$ = the directed distance from the x-axis.

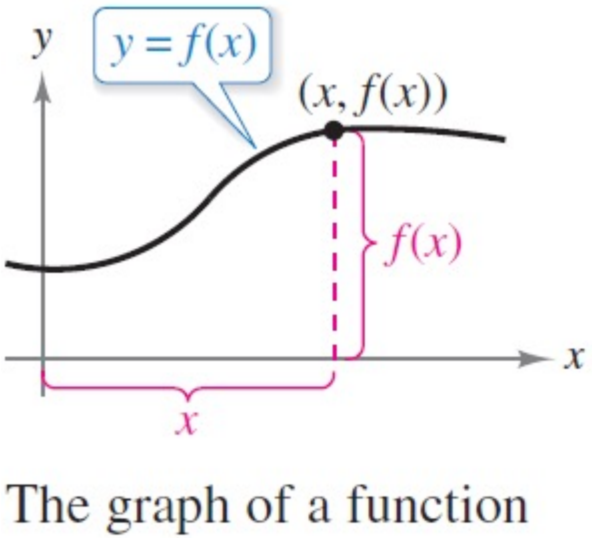


Figure P.25

The Graph of a Function

A vertical line can intersect the graph of a function of x at most *once*.

This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x.

That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point.

The Graph of a Function

For example, in Figure P.26(a), you can see that the graph does not define *y* as a function of *x* because a vertical line intersects the graph twice, whereas in Figures P.26(b) and (c), the graphs do define *y* as a function of *x*.

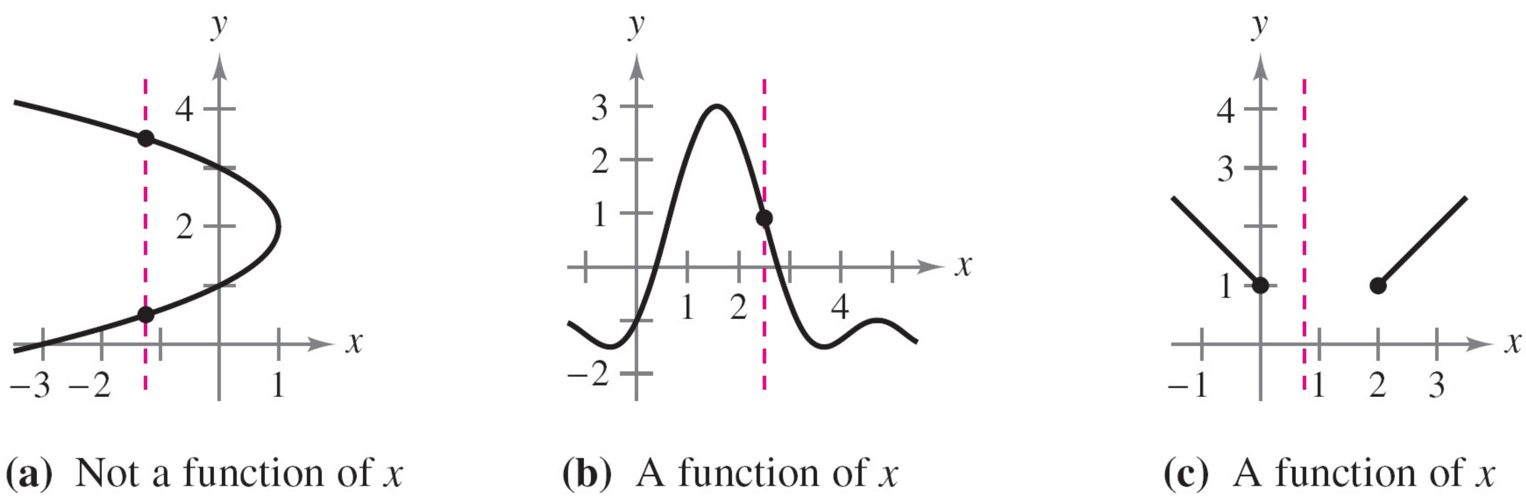
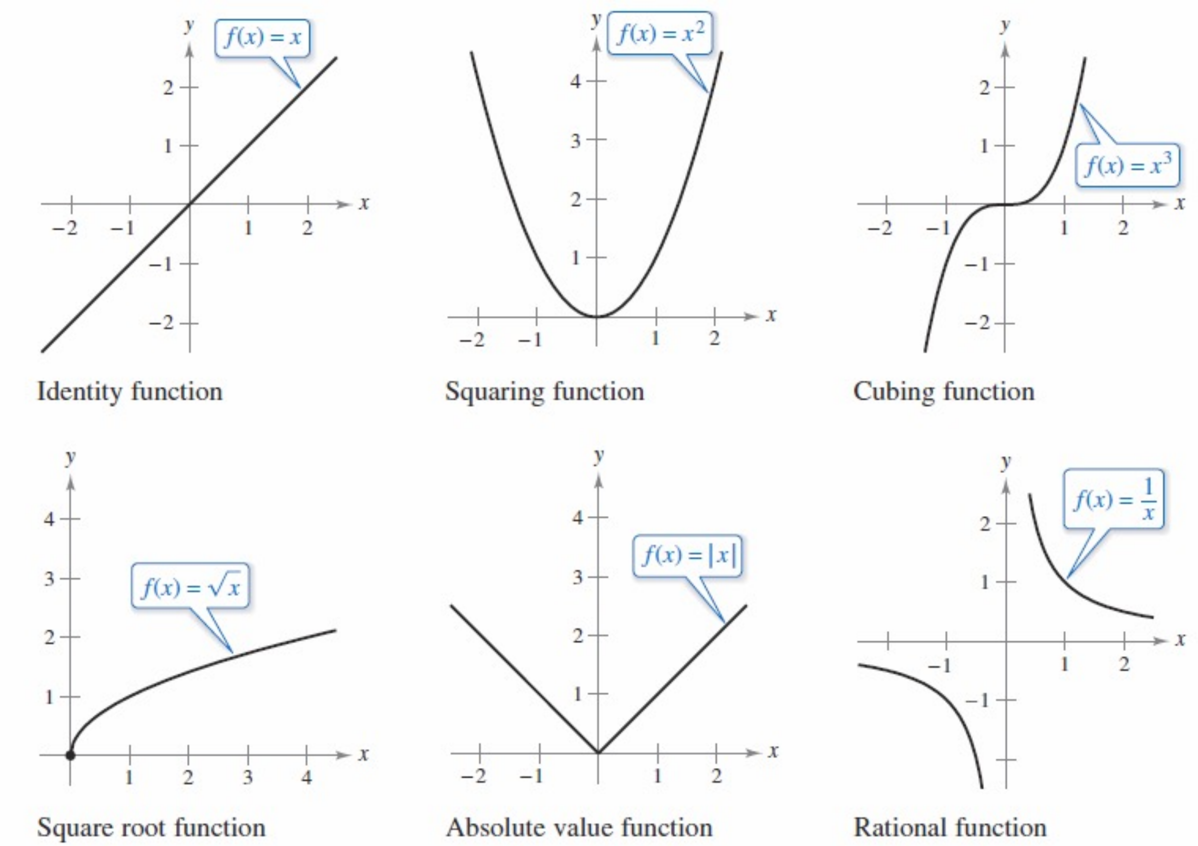


Figure P.26

The Graph of a Function

Figure P.27 shows the graphs of six basic functions.



The graphs of six basic functions

Figure P.27

Transformations of Functions

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure P.28.

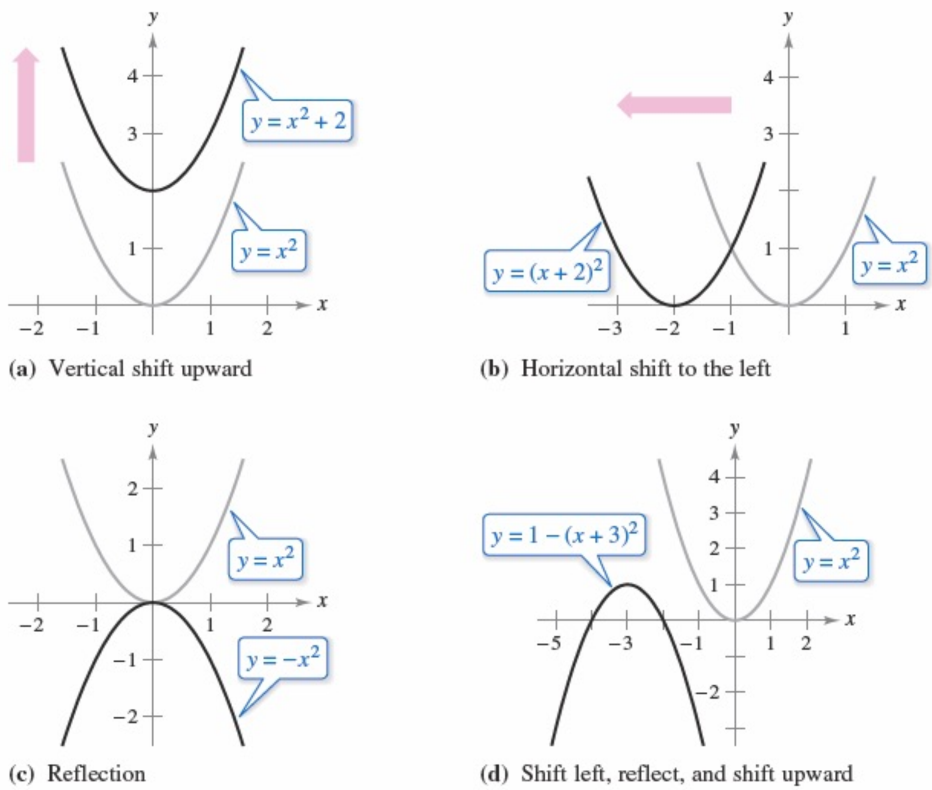


Figure P.28

Transformations of Functions

Each of the graphs in Figure P.28 is a **transformation** of the graph of $y = x^2$.

The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections.

Function notation lends itself well to describing transformations of graphs in the plane.

Transformations of Functions

For instance, using

$$f(x) = x^2 \qquad \text{Original function}$$

as the original function, the transformations shown in Figure P.28 can be represented by these equations.

a. $y = f(x) + 2$ Vertical shift up two units

b. $y = f(x + 2)$ Horizontal shift to the left two units

c. $y = -f(x)$ Reflection about the x-axis

d. $y = -f(x + 3) + 1$ Shift left three units, reflect about the x-axis, and shift up one unit

Transformations of Functions

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Classifications and Combinations of Functions

Classifications and Combinations of Functions

By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.**

Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)

2. Trigonometric functions (sine, cosine, tangent, and so on)

3. Exponential and logarithmic functions

Classifications and Combinations of Functions

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer.

The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function.

If $a_n \neq 0$, then n is the **degree** of the polynomial function.

Classifications and Combinations of Functions

The zero polynomial $f(x) = 0$ is not assigned a degree.

It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, these simpler forms are often used. (Note that $a \neq 0$.)

Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left.

Classifications and Combinations of Functions

Whether the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29.

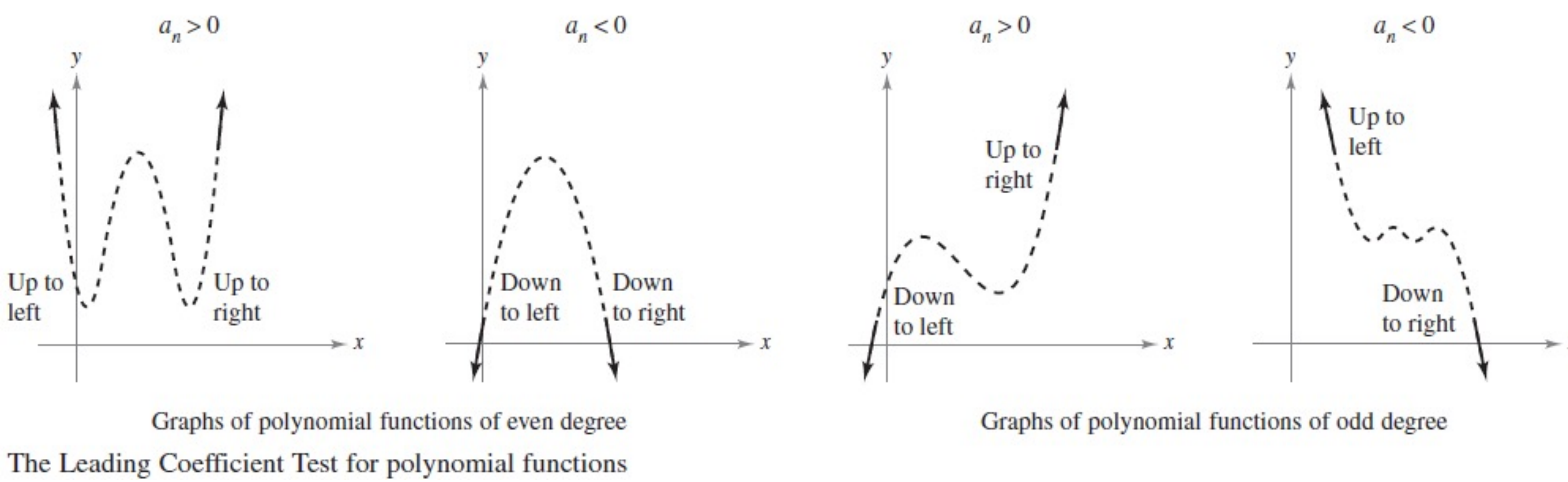


Figure P.29

Classifications and Combinations of Functions

Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.

Classifications and Combinations of Functions

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function *f* is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where *p*(*x*) and *q*(*x*) are polynomials.

Classifications and Combinations of Functions

Polynomial functions and rational functions are examples of **algebraic functions.**

An algebraic function of *x* is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n.

For example, $f(x) = \sqrt{x + 1}$ is algebraic. Functions that are not algebraic are **transcendental.**

For instance, the trigonometric functions are transcendental.

Classifications and Combinations of Functions

Two functions can be combined in various ways to create new functions. For example, given

$$f(x) = 2x - 3 \text{ and } g(x) = x^2 + 1,$$

you can form the functions shown.

$$(f + g)(x) = f(x) + g(x) = (2x - 3) + (x^2 + 1) \qquad \text{Sum}$$

$$(f - g)(x) = f(x) - g(x) = (2x - 3) - (x^2 + 1) \qquad \text{Difference}$$

$$(fg)(x) = f(x)g(x) = (2x - 3)(x^2 + 1) \qquad \text{Product}$$

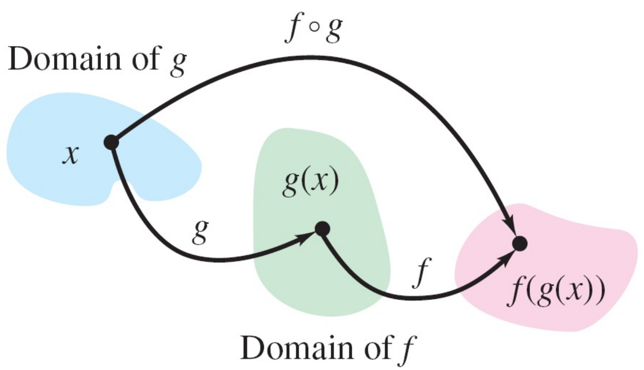
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \qquad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**

Classifications and Combinations of Functions

Definition of Composite Function

Let f and g be functions. The function $(f \circ g)(x) = f(g(x))$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).



The domain of the composite function $f \circ g$

Figure P.30

The composite of *f* with *g* is generally not the same as the composite of *g* with *f.*

Example 4 – *Finding Composite Functions*

For $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution:

$$
\begin{aligned}
\textbf{a.}\ (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\
&= f(x^2 + 1) && \text{Substitute } x^2 + 1 \text{ for } g(x). \\
&= 2(x^2 + 1) - 3 && \text{Definition of } f(x) \\
&= 2x^2 - 1 && \text{Simplify.}
\end{aligned}
$$

Example 4 – *Solution*

cont'd

b. $(g \circ f)x = g(f(x))$ Definition of $g \circ f$

$= g(2x - 3)$ Substitute $2x - 3$ for $f(x)$.

$= (2x - 3)^2 + 1$ Definition of $g(x)$

$= 4x^2 - 12x + 10$ Simplify.

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

Classifications and Combinations of Functions

An x-intercept of a graph is defined to be a point (a, 0) at which the graph crosses the x-axis. If the graph represents a function f, then the number a is a **zero** of f.

In other words, *the zeros of a function f are the solutions of the equation* $f(x) = 0$. For example, the function

$$f(x) = x - 4$$

has a zero at $x = 4$ because $f(4) = 0$.

Classifications and Combinations of Functions

In the terminology of functions, a function $y = f(x)$ is **even** when its graph is symmetric with respect to the y-axis, and is **odd** when its graph is symmetric with respect to the origin.

The symmetry tests yield the following test for even and odd functions.

Test for Even and Odd Functions

The function $y = f(x)$ is **even** when

$$f(-x) = f(x).$$

The function $y = f(x)$ is **odd** when

$$f(-x) = -f(x).$$

Example 5 – *Even and Odd Functions and Zeros of Functions*

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = \dfrac{1}{x^2}$ **c.** $h(x) = -x^2 - x - 1$

Solution:

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

Example 5 – *Solution* cont'd

The zeros of *f* are

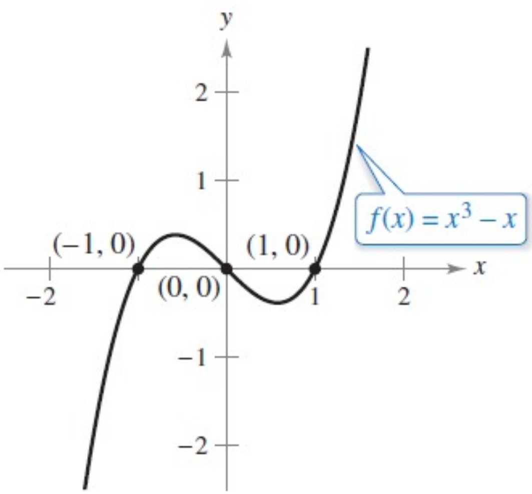
$$x^3 - x = 0 \qquad \text{Let } f(x) = 0.$$

$$x(x^2 - 1) = 0 \qquad \text{Factor.}$$

$$x(x - 1)(x + 1) = 0 \qquad \text{Factor.}$$

$$x = 0, 1, -1. \qquad \text{Zeros of } f$$

See Figure P.31(a).



(a) Odd function

Figure P.31

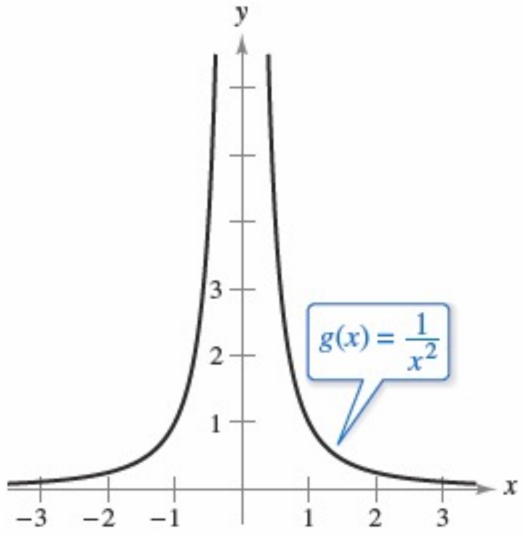
Example 5 – *Solution*

cont'd

b. This function is even because

$$g(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = g(x).$$

This function does not have zeros because $1/x^2$ is positive for all x in the domain, as shown in Figure P.31(b).



(b) Even function

Figure P.31

Example 5 – *Solution*

cont'd

c. Substituting $-x$ for x produces

$$h(-x) = -(-x)^2 - (-x) - 1 = -x^2 + x - 1.$$

Because $h(x) = -x^2 - x - 1$ and $-h(x) = x^2 + x + 1$, we can conclude that

$$h(-x) \neq h(x)$$ Function is *not* even.

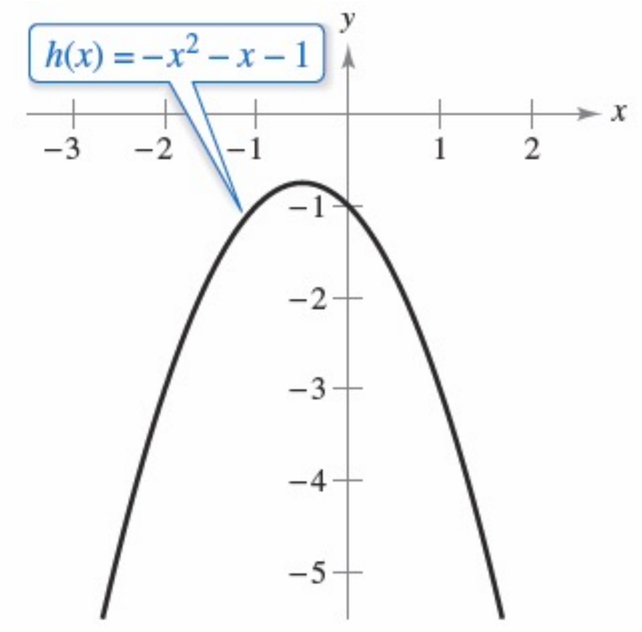
and

$$h(-x) \neq -h(x).$$ Function is *not* odd.

Example 5 – *Solution*

cont'd

So, the function is neither even nor odd. This function does not have zeros because $-x^2 - x - 1$ is negative for all x, as shown in Figure P.31(c).



(c) Neither even nor odd

Figure P.31